# Water Newtonian noise for KAGRA

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We consider Newtonian noise from a water channel and a water fall and estimate how large these Newtonian noises are to conclude whether it finally affects KAGRA sensitivity or not.

## I. NEWTONIAN NOISE FROM A WATER CHANNEL

### A. Noise power spectrum

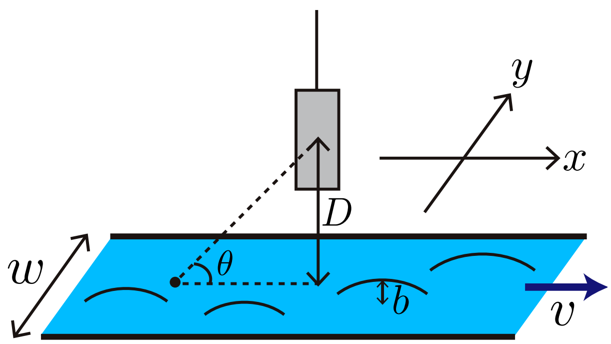


Figure 1. Schematic views of Newtonian force exerted on a mirror from a water channel.

We consider a turbulent water flow along a channel near a mirror. Suppose that $D$ is closest distance between a mirror and the water channel, $w$ is width of the water channel, $v$ is average velocity of the water flow, and $\delta v$ is velocity fluctuation from the average. The time scale of the water flow relevant to Newtonian noise is $D/v$, while the time scale of water turbulence is $w/\delta v$. If the condition

$$\frac{D}{v} \ll \frac{w}{\delta v} \,, \tag{1}$$

is satisfied, the spacial pattern of the water surface does not change during the relevant period of interaction with the mirror. Indeed, for $D = 2\,\mathrm{m}$, $w = 0.4\,\mathrm{m}$, $v = 1\,\mathrm{m/s}$, and $\delta v = 0.03\,\mathrm{m/s}$ [1], the above condition holds well. Then the water flow is treated as a fixed water-surface pattern flowing with constant velocity (Yanbei Chen's model 2 [2]).

Denoting the coordinates of a water fragment with density $\rho$ by $x$ along water flow and $y$ across the channel and water surface height by $b(t, x, y)$, the Newtonian force exerted on the mirror is

$$\begin{aligned} F(t) &= \int_{-\infty}^{\infty} dx \int dy \frac{G\rho b(t,x,y)}{D^2+x^2+y^2} \cos\theta \\ &= G\rho \int_{-\infty}^{\infty} dx \int dy \frac{b(x,y)x}{(D^2+x^2+y^2)^{3/2}} \\ &= -G\rho \int_{-\infty}^{\infty} dx \int dy \frac{\partial b(t,x,y)}{\partial x} \frac{1}{(D^2+x^2+y^2)^{1/2}} \\ &\approx -G\rho w \int_{-\infty}^{\infty} dx \frac{\partial b(t,x)}{\partial x} \frac{1}{(D^2+x^2)^{1/2}} \,. \end{aligned} \tag{2}$$

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[1] $\delta v$ is typically less than 3% of $v$ for an open channel [1].

At the last line, we assumed $x \gg y$ and ignored water height fluctuations in the $y$ direction just for simplicity. Actually this slightly overestimates the noise. The Fourier transform is

$$
\begin{aligned}
\tilde{F}(\Omega) &= ikG\rho w\tilde{b}(\Omega,k)\int_{-\infty}^{\infty} dx\frac{e^{-ikx}}{(D^2+x^2)^{1/2}} \\
&= 2ikG\rho w\tilde{b}(\Omega,k)\int_{0}^{\infty} dx\frac{\cos(kx)}{(D^2+x^2)^{1/2}} \\
&= 2ikG\rho w\tilde{b}(\Omega,k)\int_{0}^{\infty} dx'\frac{\cos(kDx')}{\{1+(x')^2\}^{1/2}} \\
&= 2ikG\rho w\tilde{b}(\Omega,k)K_0(kD)\ ,
\end{aligned} \tag{3}
$$

where $K_0$ is the 0th order modified Bessel function of the second kind. Since $v = \Omega k$ in the $x$ direction, the Fourier component is expressed as a function of $\Omega$ and $v$:

$$
\tilde{F}(\Omega) = \frac{2iG\rho w\Omega}{v}\,\tilde{b}\,(\Omega)\,K_0\left(\frac{\Omega D}{v}\right)\ . \tag{4}
$$

Defining the power spectral densities by

$$
\langle\tilde{F}(\Omega)\tilde{F}^*(\Omega')\rangle = S_F(\Omega)\delta(\Omega-\Omega')\ , \tag{5}
$$

$$
\langle\tilde{b}(\Omega)\tilde{b}^*(\Omega')\rangle = S_b(\Omega)\delta(\Omega-\Omega')\ , \tag{6}
$$

we have

$$
\sqrt{S_F(\Omega)} = \frac{2G\rho w\Omega}{v}K_0\left(\frac{\Omega D}{v}\right)\sqrt{S_b(\Omega)}\ . \tag{7}
$$

In terms of strain amplitude, it is

$$
\begin{aligned}
\sqrt{S_h(\Omega)} &= \frac{2G\rho w}{\Omega L v}K_0\left(\frac{\Omega D}{v}\right)\sqrt{S_b(\Omega)} \\
&= \frac{2G\rho w}{\Omega v}K_0\left(\frac{\Omega D}{v}\right)\frac{\sqrt{S_{b,1\mathrm{Hz}}}}{L}\left(\frac{f}{1\,\mathrm{Hz}}\right)^{\beta/2}\ .
\end{aligned} \tag{8}
$$

The modified Bessel function $K_0(q)$ is shown in Fig. 2. This function gives exponential suppression at $q \gg 1$. That is, the noise is drastically suppressed at high frequencies, large distance from the mirror, or small water velocity. Some examples of the power spectrum are shown in Figs. 3 - 5.

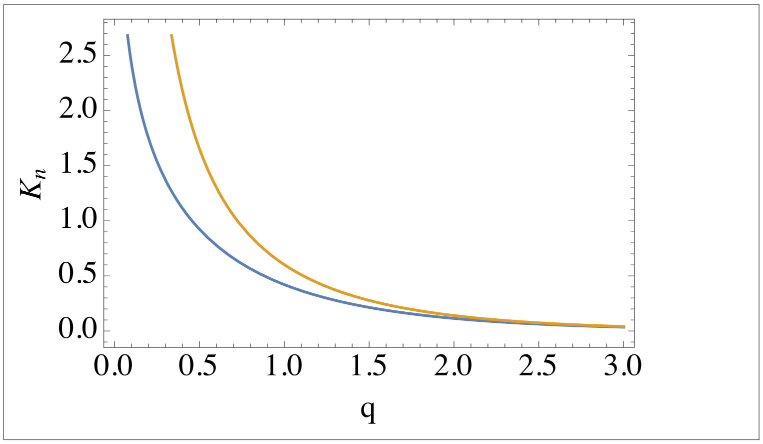


Figure 2. Modified Bessel function of the second kind, $K_0(q)$ (blue) and $K_1(q)$ (orange).

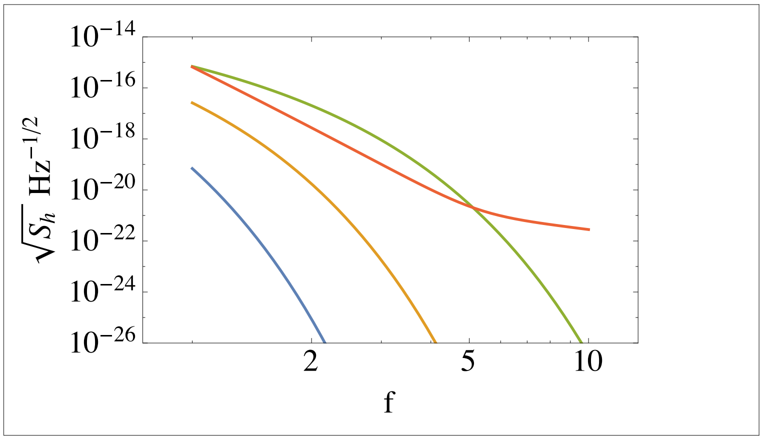


Figure 3. Power spectra of Newtonian noise from a water channel as a function of frequency $f$ ($\Omega = 2\pi f$) for different water flow velocities $v = 1\,\mathrm{m}/s$ (blue), $2\,\mathrm{m}/s$ (orange), and $5\,\mathrm{m}/s$ (green). Other parameters are set to $D = 2\,\mathrm{m}$, $w = 0.4\,\mathrm{m}$, $\rho = 1\,\mathrm{g/cm^3}$, $L = 3\,\mathrm{km}$, $\beta = 0$, $S_{b,1\mathrm{Hz}} = 1\,\mathrm{cm^2/Hz}$. The red line is KAGRA design noise curve [3].

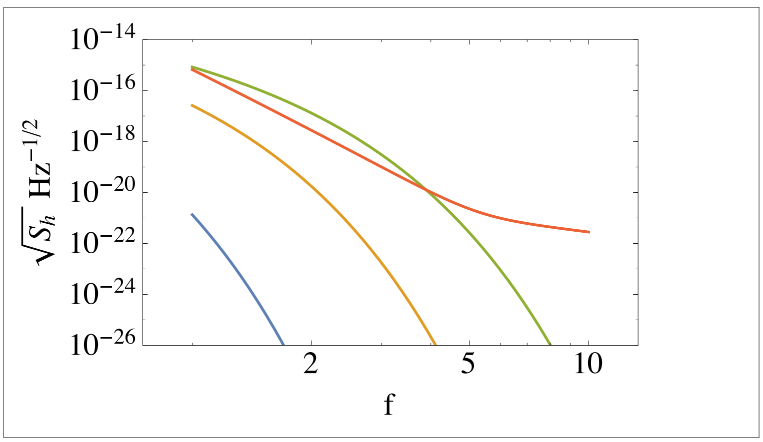


Figure 4. Power spectra of Newtonian noise from a water channel as a function of frequency $f$ ($\Omega = 2\pi f$) for different distance to the mirror $D = 5\,\mathrm{m}$ (blue), $2\,\mathrm{m}$ (orange), and $1\,\mathrm{m}$ (green). Other parameters are set to $w = 0.4\,\mathrm{m}$, $\rho = 1\,\mathrm{g/cm^3}$, $L = 3\,\mathrm{km}$, $\beta = 0$, $S_{b,1\mathrm{Hz}} = 1\,\mathrm{cm^2/Hz}$, $v = 2\,\mathrm{m}/s$. The red line is KAGRA design noise curve [3].

### B. Water velocity in an open channel

For a long wave whose wavelength is much larger than depth and width of a channel, water flow velocity is given in Sec. 13 of [4]

$$v = \sqrt{\frac{gS_0}{w}}\,. \tag{9}$$

Here $g$ is gravity acceleration, $S_0$ is water flow cross-section, and $w$ is width of the channel at the water surface. For $g = 9.8\,\mathrm{m/s^2}$, $S_0 = \pi w^2/8$ (half pipe), and $w = 0.4\,\mathrm{m}$, the velocity is $v = 1.24\,\mathrm{m/s}$.

For a short wave whose wavelength is smaller than depth and width of a channel, water flow velocity is given in Sec. 12 of [4]

$$v = \frac{1}{2}\sqrt{\frac{g\lambda}{2\pi}}\,. \tag{10}$$

Since $\lambda \ll w$, the velocity of a short wave should be smaller than that of a long wave. Therefore, consequent Newtonian noise caused by short waves is subdominant.

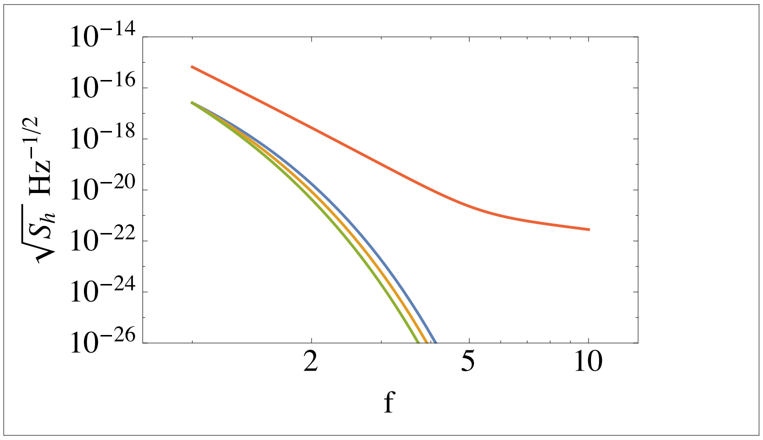


Figure 5. Power spectra of Newtonian noise from a water channel as a function of frequency $f$ ($\Omega = 2\pi f$) for different spectral tilt $\beta = 0$ (blue), $-2$ (orange), and $-4$ (green). Other parameters are set to $D = 2\,\mathrm{m}$, $w = 0.4\,\mathrm{m}$, $\rho = 1\,\mathrm{g/cm^3}$, $L = 3\,\mathrm{km}$, $S_{b,1\mathrm{Hz}} = 1\,\mathrm{cm^2/Hz}$, $v = 2\,\mathrm{m}/s$. The red line is KAGRA design noise curve [3].

## II. NEWTONIAN NOISE FROM A WATER FALL

### A. Noise power spectrum

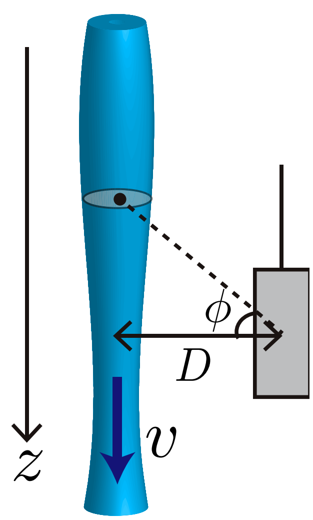


Figure 6. Schematic views of Newtonian force exerted on a mirror from a water fall.

The derivation of noise power spectral density is similar to that in the previous section. But in this case, water falls from a ceiling to a floor, being accelerated by gravity and changing velocity. For simplicity we assume that velocity is constant during the fall and take maximum velocity as the constant velocity. In this sense, we overestimate the noise, but our estimation should be conservative. Under this assumption the computation is parallel to that in the previous section.

Denoting the coordinates of a water fragment with density $\rho$ by $z$ along the vertical (water-falling) direction and

$(D, y)$ on the horizontal plane and water cross-section by $A(t, z)$, the Newtonian force exerted on the mirror is

$$\begin{aligned} F(t) &= \int_{-\infty}^{\infty} dz \frac{G\rho A(t,z)}{D^2+y^2+z^2} \cos\phi \\ &= G\rho \int_{-\infty}^{\infty} dz \frac{A(t,z)(D^2+y^2)^{1/2}}{(D^2+y^2+z^2)^{3/2}} \\ &\approx G\rho D \int_{-\infty}^{\infty} dz \frac{A(t,z)}{(D^2+z^2)^{3/2}} \ . \end{aligned} \tag{11}$$

At the last line, we assumed $y \ll D$ and dropped the $y$ coordinate. This always underestimates actual distance from a mirror and overestimates corresponding Newtonian noise. The Fourier transform is

$$\begin{aligned} \tilde{F}(\Omega) &= G\rho D \tilde{A}(\Omega, k) \int_{-\infty}^{\infty} dz \frac{e^{-ikz}}{(D^2+z^2)^{3/2}} \\ &= 2G\rho D \tilde{A}(\Omega, k) \int_{0}^{\infty} dz \frac{\cos(kz)}{(D^2+z^2)^{3/2}} \\ &= \frac{2G\rho \tilde{A}(\Omega, k)}{D} \int_{0}^{\infty} dz' \frac{\cos(kDz')}{\{1+(z')^2\}^{3/2}} \\ &= 2kG\rho \tilde{A}(\Omega, k) K_1(kD) \ , \end{aligned} \tag{12}$$

where $K_1$ is the 1st order modified Bessel function of the second kind. Since $v = \Omega k$ in the $z$ direction, the Fourier component is expressed as a function of $\Omega$ and $v$:

$$\tilde{F}(\Omega) = \frac{2G\rho\Omega}{v} \tilde{A}(\Omega) K_1 \left( \frac{\Omega D}{v} \right) \ . \tag{13}$$

Defining the power spectral densities by

$$\langle \tilde{F}(\Omega) \tilde{F}^*(\Omega') \rangle = S_F(\Omega) \delta(\Omega - \Omega') \ , \tag{14}$$

$$\langle \tilde{A}(\Omega) \tilde{A}^*(\Omega') \rangle = S_A(\Omega) \delta(\Omega - \Omega') \ , \tag{15}$$

we have

$$\sqrt{S_F(\Omega)} = \frac{2G\rho\Omega}{v} K_1 \left( \frac{\Omega D}{v} \right) \sqrt{S_A(\Omega)} \ . \tag{16}$$

In terms of strain amplitude, it is

$$\begin{aligned} \sqrt{S_h(\Omega)} &= \frac{2G\rho}{\Omega L v} K_1 \left( \frac{\Omega D}{v} \right) \sqrt{S_A(\Omega)} \\ &= \frac{2G\rho L}{\Omega v} K_1 \left( \frac{\Omega D}{v} \right) \frac{\sqrt{S_{A,1\mathrm{Hz}}}}{L^2} \left( \frac{f}{1\,\mathrm{Hz}} \right)^{\alpha/2} \ . \end{aligned} \tag{17}$$

The modified Bessel function $K_1(q)$ is shown in Fig. 2. This function gives exponential suppression at $q \gg 1$. That is, the noise is drastically suppressed at high frequency, large distance from the mirror, or small water velocity. Some examples of the power spectrum are shown in Figs. 7 - 9.

### B. Velocity of water falling in gravity

In case of a water sphere with diameter from a few to several cm in the air with room temperature, inertial resistance force balances with gravitational force and gives terminal velocity

$$v_t = 4\sqrt{\frac{rg}{3}} \ , \tag{18}$$

where $r$ is radius of the sphere. For $r = 1\,\mathrm{cm}$, $3\,\mathrm{cm}$, and $10\,\mathrm{cm}$, the final velocity is $0.72\,\mathrm{m/s}$, $1.3\,\mathrm{m/s}$, and $2.3\,\mathrm{m/s}$, respectively.

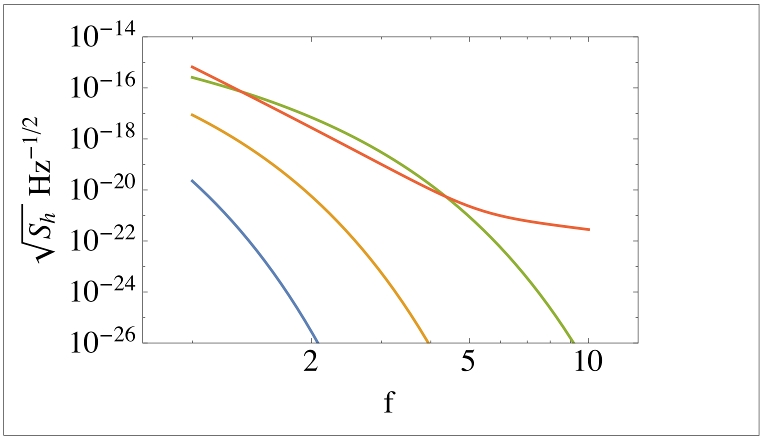


Figure 7. Power spectra of Newtonian noise from a water fall as a function of frequency $f$ ($\Omega = 2\pi f$) for different water flow velocities $v = 1\,\mathrm{m}/s$ (blue), $2\,\mathrm{m}/s$ (orange), and $5\,\mathrm{m}/s$ (green). Other parameters are set to $D = 2\,\mathrm{m}$, $\rho = 1\,\mathrm{g/cm^3}$, $L = 3\,\mathrm{km}$, $\alpha = 0$, $S_{A,1\mathrm{Hz}} = 5^4\,\mathrm{cm^4/Hz}$. The red line is KAGRA design noise curve [3].

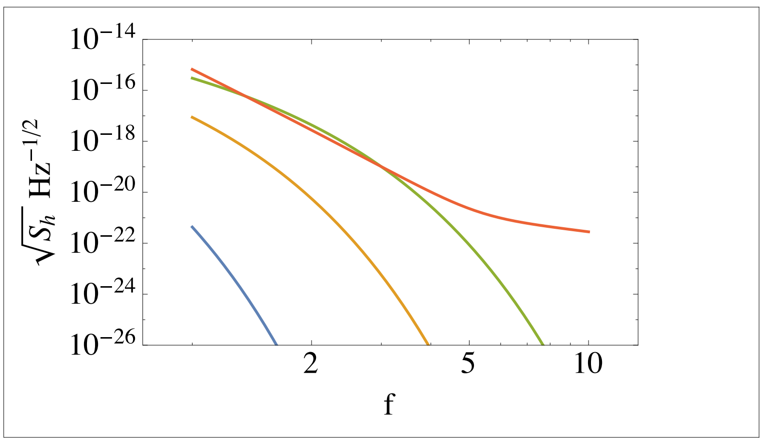


Figure 8. Power spectra of Newtonian noise from a water fall as a function of frequency $f$ ($\Omega = 2\pi f$) for different distance to the mirror $D = 5\,\mathrm{m}$ (blue), $2\,\mathrm{m}$ (orange), and $1\,\mathrm{m}$ (green). Other parameters are set to $\rho = 1\,\mathrm{g/cm^3}$, $L = 3\,\mathrm{km}$, $\alpha = 0$, $S_{A,1\mathrm{Hz}} = 5^4\,\mathrm{cm^4/Hz}$, $v = 2\,\mathrm{m}/s$. The red line is KAGRA design noise curve [3].

## III. DISCUSSION

From Figs. 10 and 11, we see that the parameter region that KAGRA sensitivity is contaminated by water Newtonian noise is an extreme case in which water velocity is so high ($v \gtrsim 10\,\mathrm{m/s}$) or water flow is so close to a mirror ($D \lesssim 1\,\mathrm{m}$). Therefore it seems that water Newtonian noise is irrelevant to KAGRA.

However, this conclusion should be taken with a caution. In the computation, we assume that water turbulence does not evolve in a time scale shorter than the interaction time scale of the water flow, i.e. Eq. (1). This condition must be checked in actual measurements of a water surface. In addition, critical parameters, $v$ and $D$, should be measured.

Finally here we considered Newtonian noise from a water flow, but water makes sounds and may contribute to KAGRA sensitivity (water acoustic noise). For example, when a water fall hit a floor and when a water flow is highly turbulent at an obstacle or a corner.

[1] (In Japanese) 禰津家久、冨永晃宏、「水理学」、朝倉書店 (2006).

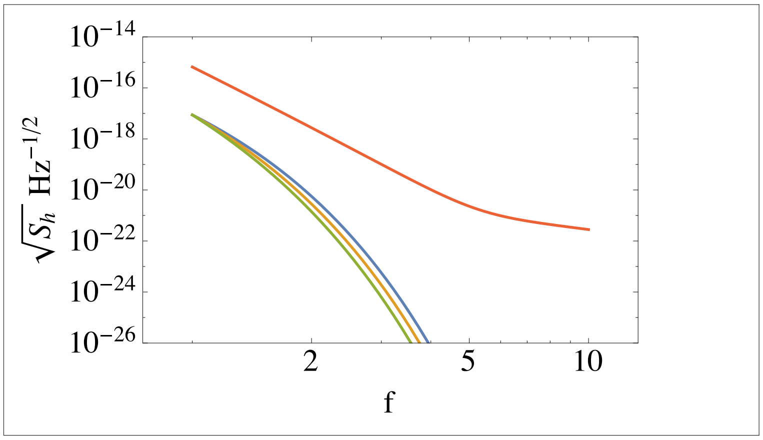


Figure 9. Power spectra of Newtonian noise from a water fall as a function of frequency $f$ ($\Omega = 2\pi f$) for different spectral tilt $\alpha = 0$ (blue), $-2$ (orange), and $-4$ (green). Other parameters are set to $D = 2\,\mathrm{m}$, $\rho = 1\,\mathrm{g/cm^3}$, $L = 3\,\mathrm{km}$, $S_{A,1\mathrm{Hz}} = 5^4\,\mathrm{cm^2/Hz}$, $v = 2\,\mathrm{m}/s$. The red line is KAGRA design noise curve [3].

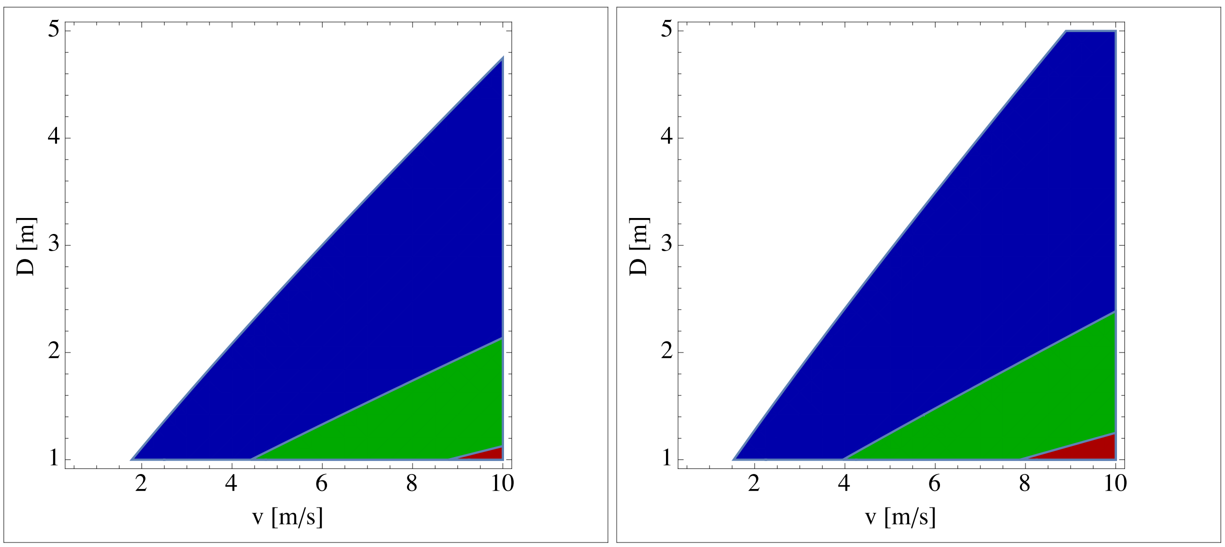


Figure 10. Parameter region on $(v, D)$ plane where water Newtonian noise from a water channel exceeds KAGRA design noise at a certain frequency: $f = 3\,\mathrm{Hz}$ (blue), $10\,\mathrm{Hz}$ (green), and $20\,\mathrm{Hz}$ (red). Other parameters are set to $w = 0.4\,\mathrm{m}$, $\rho = 1\,\mathrm{g/cm^3}$, $L = 3\,\mathrm{km}$, $\beta = 0$. $S_{b,1\mathrm{Hz}} = 1\,\mathrm{cm^2/Hz}$ (left) and $S_{b,1\mathrm{Hz}} = 25\,\mathrm{cm^2/Hz}$ (right).

[2] Yanbei Chen, internal document ”Gravity Gradient Noise from Water”.
[3] A. Manzotti and A. Dietz, arXiv:1202.4031 (2012).
[4] L. D. Landau and E. M. Lifshitz, ”Fluid Mechanics”, Course of theoretical physics, Oxford: Pergamon Press (1959).

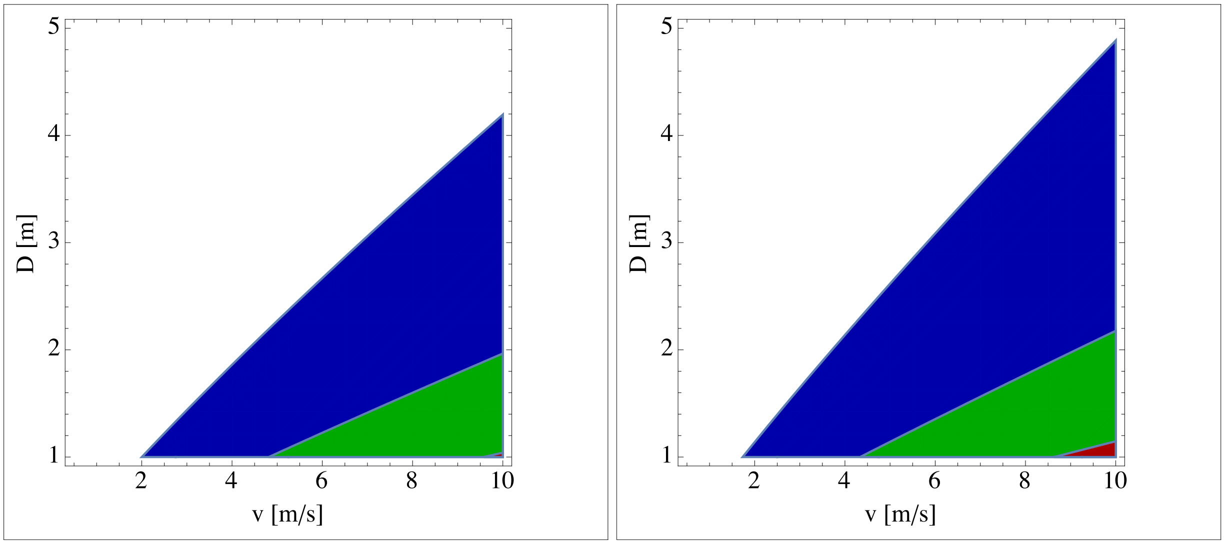


Figure 11. Parameter region on $(v, D)$ plane where water Newtonian noise from a water fall exceeds KAGRA design noise at a certain frequency: $f = 3\,\mathrm{Hz}$ (blue), $10\,\mathrm{Hz}$ (green), and $20\,\mathrm{Hz}$ (red). Other parameters are set to $\rho = 1\,\mathrm{g/cm}^3$, $L = 3\,\mathrm{km}$, $\alpha = 0$. $S_{A,1\mathrm{Hz}} = 5^4\,\mathrm{cm}^2/\mathrm{Hz}$ (left) and $S_{A,1\mathrm{Hz}} = 10^4\,\mathrm{cm}^2/\mathrm{Hz}$ (right).