Mechanical Transfer Function of a Pendulum Suspended with a Finite-Mass Wire

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Short Note

Mechanical Transfer Function of a Pendulum Suspended with a Finite-Mass Wire

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In free-mass gravitational wave detectors, pendulum suspension is one of the most important parts. In this report, the mechanical transfer functions are calculated for the pendulum comprised of a wire of uniform density and a point mass attached to the end of the wire, and their characteristics are discussed in terms of the normal-mode expansion method.

KEYWORDS: interferometric gravitational wave detector, pendulum, wire resonance, thermal noise, mechanical transfer function

In free-mass gravitational wave detectors, pendulum suspension is one of the most important parts, because its mechanical properties probably dominate the vibration isolation, the thermal noise and the performance of the mirror control systems. In particular, the transverse resonance of the suspension wire due to its finite mass has a serious effect on these factors.[1,2] In this paper, the mechanical transfer functions of the pendulum are calculated and their characteristics are discussed in terms of the normal-mode expansion method.

Here, the pendulum is treated as a wire of length l and uniform linear mass density σ with a point mass M attached to the end of the wire. The displacement of the wire u is determined by

$$\frac{\partial^2 u}{\partial t^2}-v^2\frac{\partial^2 u}{\partial z^2}=0 \quad \left(v=\sqrt{\frac{F}{\sigma}}\right), \tag{1}$$

and the equation of motion of the end mass is given by

$$M\left.\frac{\partial^2 u}{\partial t^2}\right|_{z=l}=-F\left.\frac{\partial u}{\partial z}\right|_{z=l}+f(t), \tag{2}$$

where F and f are the tension of the wire and the external force applied to the mass, respectively. When the force is $f(t)=f_0\,\mathrm{e}^{\mathrm{i}\omega t}$, the solution of eq. (1) can be expressed as

$$u(z, t)=A\,\mathrm{e}^{\mathrm{i}\omega t}\sin kz \quad \left(k=\frac{\omega}{v}\right), \tag{3}$$

because the boundary condition at the suspension point ($z=0$) is given by $x_0(t)=u(0, t)=0$. From eq. (2), the displacement of the mass, $x(t)$ is presented as

$$x(t)=u(l, t)=\frac{f_0\,\mathrm{e}^{\mathrm{i}\omega t}\sin kl}{Fk\cos kl-M\omega^2\sin kl}. \tag{4}$$

Thus, the transfer function between the force applied to the mass and its displacement, $H_{f\to x}$ can be written as

$$H_{f\to x}(\omega)=\frac{\sin(\omega l/v)}{\omega l/v}\,\frac{1}{M\omega_{\mathrm{p}}^2[\cos(\omega l/v)-v\omega/g\sin(\omega l/v)]}, \tag{5}$$

where $\omega_{\mathrm{p}}=\sqrt{g/l}$ and we assume the relation $F=Mg$. The frequency of the nth resonance mode is determined by the formula,

$$\cos(\omega_n l/v)-(v\omega_n/g)\sin(\omega_n l/v)=0. \tag{6}$$

While the lowest resonance mode ($n=0$) represents pendular motion ($\omega_0\approx\omega_{\mathrm{p}}$), higher ones ($n\geq 1$) are the violin modes. The resonant frequency of the nth violin mode can be approximately given by[1]

$$\omega_n\approx n\pi\omega_{\mathrm{p}}\sqrt{\frac{M}{m}}, \tag{7}$$

where $m(=\sigma l)$ is the mass of the wire. The transfer function between the displacement of the suspension point and the mass displacement, $H_{x_0\to x}$ is given by[1]

$$H_{x_0\to x}(\omega)=\frac{1}{\cos(\omega l/v)-(v\omega/g)\sin(\omega l/v)}. \tag{8}$$

The significant difference between eq. (5) and eq. (8) is the factor $\sin(\omega l/v)/(\omega l/v)$ in eq. (5); this factor becomes very small near the resonances as

$$\left|\frac{\sin(\omega_n l/v)}{\omega_n l/v}\right|\approx\left(\frac{1}{n\pi}\right)^2\frac{m}{M}. \tag{9}$$

In the above derivation, however, the behavior of these functions at the resonances is rather ambiguous, because the quality factor Q of the resonance is infinite. In order to account for the loss, the normal-mode expansion method is very effective and it elucidates the effects of the higher resonances.

We can define the eigen function of the nth normal mode by

$$w_n(z)=\frac{\sin k_n z}{\sin k_n l}, \tag{10}$$

and show the orthogonal condition as[2]

$$M+\sigma\int_0^l w_n(z)w_m(z)\,\mathrm{d}z=\delta_{nm}\mu_n. \tag{11}$$

Here, μ_n is the reduced mass of the nth mode calculated as

$$\mu_n=M+\sigma\int_0^l w_n(z)^2\,\mathrm{d}z, \tag{12}$$

according to the general definition of the reduced mass.[3] Using eq. (6), we obtain

$$\mu_n=\frac{M}{2}\left[1+\frac{1}{\cos^2(k_n l)}\left(\frac{\omega_n}{\omega_{\mathrm{p}}}\right)^2\right]. \tag{13}$$

By means of eqs. (10)–(12), eq. (5) can be expressed as

$$H_{f\to x}=\sum_{n=0}^{\infty}\frac{1}{\mu_n(\omega_n^2-\omega^2)}; \tag{14}$$

the validity of this expansion was confirmed by a numerical calculation for several frequencies. The loss of the wires can be taken into consideration by introducing a complex spring constant as[2]

$$\omega_n^2 \to \omega_n^2[1+\mathrm{i}\phi_n(\omega)].$$

As a result, $H_{f\to x}$ can be written as

$$H_{f\to x}=\sum_{n=0}^{\infty}\frac{1}{\mu_n(\omega_n^2[1+\mathrm{i}\phi_n(\omega)]-\omega^2)}, \tag{15}$$

in this model, the Q-value of the resonance is given by $1/\phi_n(\omega_n)$. To estimate the contribution from the violin modes, we consider the case that $\omega\sim\omega_n$ then obtain the ratio of the magnitude of the nth mode to that of the fundamental one as $Q_nM/\mu_n\sim Q_n(\omega_\mathrm{p}/\omega_n)^2/2$. Since the factor $(\omega_n/\omega_\mathrm{p})^2$ will reach an order of 10^6, it is hard to find the resonances in $H_{f\to x}$.

On the other hand, eq. (8) can be written as

$$\begin{aligned} H_{x_0\to x}&=\sum_{n=0}^{\infty}\frac{M}{\mu_n\cos(k_nl)}\frac{\omega_n^2[1+\mathrm{i}\phi_n(\omega)]}{\omega_n^2[1+\mathrm{i}\phi_n(\omega)]-\omega^2}\\ &\approx\frac{\omega_\mathrm{p}^2[1+\mathrm{i}\phi_0(\omega)]}{\omega_\mathrm{p}^2[1+\mathrm{i}\phi_0(\omega)]-\omega^2}\\ &\quad+2\sum_{n=1}^{\infty}\frac{(-1)^n\omega_\mathrm{p}^2[1+\mathrm{i}\phi_n(\omega)]}{\omega_n^2[1+\mathrm{i}\phi_n(\omega)]-\omega^2}, \end{aligned}\tag{16}$$

where the loss has already been considered. The factor $(-1)^n$ represents the fact that the phase of $H_{x_0\to x}$ is delayed by π at every resonance; in contrast, eq. (14) shows that the phase delay of $H_{f\to x}$ does not exceed π. From eq. (16), one finds that the magnitude of the nth mode is twice as large as that of the pendulum mode when $\omega\gg\omega_n$, but is canceled by the $(n-1)$th term owing to the phase delay mentioned above. For the case where $\omega\ll\omega_n$, this term can be neglected. Therefore, it is not necessary to consider the contributions from the violin mode whose resonant frequencies are far from ω. However, when ω is in the vicinity of the resonant frequency, the effect due to the resonance may appear even if Q is on the order of unity.

Finally, the thermal noise of the pendulum is considered. The noise spectrum of the end mass is written as

$$\langle x_\mathrm{B}(\omega)^2\rangle=\frac{4k_\mathrm{B}T}{\omega}\sum_{n=0}^{\infty}\frac{\phi_n(\omega)\omega_n^2}{\mu_n[(\omega_n^2-\omega^2)^2+\phi_n(\omega)^2\omega_n^4]}, \tag{17}$$

according to the calculation by Saulson.[2] When $\omega_0\ll\omega<\omega_1$, eq. (17) is approximated as

$$\begin{aligned}\langle x_\mathrm{B}(\omega)^2\rangle&\approx\frac{4k_\mathrm{B}T}{\omega}\left(\frac{\phi_0(\omega)\omega_0^2}{M\omega^4}+\sum_{n=0}^{\infty}\frac{\phi_n(\omega)}{\mu_n\omega_n^2}\right)\\ &\approx\frac{4k_\mathrm{B}T\omega_\mathrm{p}^2}{M\omega^5}\left(\phi_0(\omega)+2\left(\frac{\omega}{\omega_1}\right)^4\sum_{n=1}^{\infty}\frac{\phi_n(\omega)}{n^4}\right).\end{aligned}\tag{18}$$

When ω becomes the same order of magnitude of ω_1, the second term is not negligible if ϕ_n is larger than ϕ_0. Further, steep peaks appear at the resonances; the maximum value is proportional to Q_n. By the present techniques, it is not easy to measure the thermal noise in the frequency range far from the resonance. Therefore, the frequency dependence of ϕ has been measured by observing the values of Q of several wire-resonance modes.[4] It has been also proposed that the information about ϕ of the pendulum can be obtained by measuring the phase of the transfer function $H_{f\to x}$ with precision on the order of 10^{-8}.*

Although the effects of the violin modes in $H_{f\to x}$ are not large, those in $H_{x_0\to x}$ are rather serious. Since the resonances and expected signals of gravitational waves reside in the same frequency range from several hundred Hz to a few kHz, the peaks due to the resonances are not preferable. To reduce these effects, ω_n should be as high as possible.[5] Therefore, a wire of low mass density and high tensile strength is desirable.[6] So far as the characteristics of the transfer functions are concerned, low values of Q_n seem to be preferable. While the thermal noise of the pendulum is considered to be one of the major noise sources at low frequencies in the full scale interferometer,[2] the shot noise of the laser light is regarded as the dominant noise source at the frequency range where the resonance peaks appear. In such a detector, if noise due to high-Q resonance emerges beyond the shot noise level, the electric cooling techniques for a mechanical resonator[7,8] are likely to be useful in damping such peaks without increasing the thermal noise.

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